Action-Angle Variables

(L/L ⇒ canonical variables)

Key concept: $\oint p\,dq$

Phase Space Circulation

⟺ Poincare – Cartan Invariant

$\lambda = \lambda(t)$, $E = E(t)$

$E = \text{const.}$, $\lambda = \text{const.}$, closed system

Canonical Transformations	Adiabatic Invariants ⇒	Action-Angle Variables
⇒ specify transformation rules	i.e. $\oint_{E,\lambda} p\,dq \Rightarrow$ COM with slow parametric variation	{circulation as variable (momentum)}
	{circulation as const motion}	⇒ integrability, {phase space geometry, resonance}

so

Action-Angle variables

⇒ seek variables (i.e. C.T.: $p, q \to \underset{P}{I}, \underset{Q}{\Theta}$) s/t:

$$H = H(I), \quad \text{so} \quad \dot{I} = 0, \quad \dot{\Theta} = \frac{\partial H}{\partial I}$$

[i.e. C.T. to conserved momentum, cyclic coordinate $\quad \Theta = \omega t + \Theta_0$.

⇒ C.T. is equivalent to integration of system. {A/A are variables on which system integrated}

→ crudely : integrate via new variables s/t $I \to$ 'generalized radius'
$\theta \to$ " " angle

so

$$p, q \to \theta, I$$

$$H(p, q) \to H'(I) \qquad \dot{I} = 0, \quad \dot{\theta} = \omega$$

C.T. : independent variables q, I (q, P)

$\Rightarrow$ Type II : $F_2 = F_2(q, P)$

so $p = \dfrac{\partial F_2}{\partial q}$, $Q = \dfrac{\partial F_2}{\partial P}$

$$\Rightarrow \quad p = \frac{\partial F_2}{\partial q}, \quad \theta = \frac{\partial F_2}{\partial I}$$

but $p = \dfrac{\partial F_2}{\partial q}$ equiv. to $p = \dfrac{\partial S}{\partial q}$ from H-J theory

(always, for Type II)

so can write in terms action as generating function, i.e.

$$F_2(q, P) = F_2(q, I) = S(q, I).$$

so $\quad \Theta = \dfrac{\partial S_0}{\partial I}, \quad p = \dfrac{\partial S_0}{\partial q}$

Now further:

$S_0 = S_0(q, I)$ indep. time; i.e. $\lambda = \lambda(\cancel{t}) = \text{const.}$

and $H(q, p) \Rightarrow \underset{= E(I)}{H(I)}$, with Θ cyclic

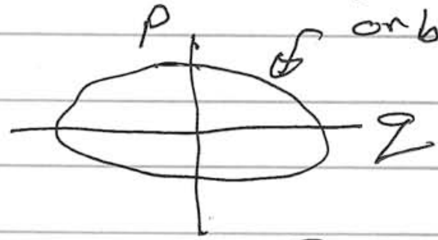
in new variables $\Rightarrow$ EOM:

$$\Rightarrow \quad \dot{I} = -\frac{\partial H}{\partial \Theta} = 0, \qquad \dot{\Theta} = \frac{\partial H}{\partial I} = \underbrace{\omega(I)}_{\text{angular frequency}}.$$

i.e. I and E constant.

Contrast: Adiabatic invariants $\Rightarrow$
$I \sim$ const, E evolves as ω evolves

Now,



$$I = \underbrace{\oint p \, dq}_{\text{1 circuit circulation}} \frac{1}{2\pi} = \underbrace{\int dp \, dq}_{\text{phase volume}}$$

another way:

$$S_0 = S_0(q, I) \quad , \quad \begin{array}{l} p = \partial S_0 / \partial q \\ \theta = \partial S_0 / \partial I \end{array}$$

so

$$\frac{d\theta}{dq} = \frac{\partial}{\partial q}\frac{\partial S}{\partial I} = \frac{\partial}{\partial I}\frac{\partial S}{\partial q}$$

$$d\theta = \frac{\partial}{\partial I}\frac{\partial S}{\partial q}\, dq$$

$\Rightarrow$

$$2\pi = \frac{\partial}{\partial I}\oint \frac{\partial S}{\partial q}\, dq$$

$$= \frac{\partial}{\partial I}\oint p\, dq.$$

$$\Rightarrow \quad \boxed{I = \oint \frac{p\, dq}{2\pi} \quad \rightarrow \text{ Action Variable}}$$

$$\boxed{\dot{\theta} = \frac{\partial H}{\partial I} = \frac{\partial E(I)}{\partial I} \equiv \underset{\text{angle variable.}}{\omega(I)}}$$

$I \rightarrow$ radius

$\omega \rightarrow$ winding rate, frequency

Comparison / Contrast

Adiabatic Invariants	A-A Variables
$\lambda = \lambda(t)$, open loop	$\lambda = \lambda_0$ const, closed loop
$I = \oint_{E,\lambda} p\,dq$ ~ approx COM	$I = \oint p\,dq$ exact COM
E varies with ω, $I \sim$ const.	E, I const. $\dot{I} = 0$ is H EOM
COM for multiple scale problems	Variables in which system is integrated i.e. $\dot{I} = 0$
1 adiabatic inv. per closed cycle (i.e. mirror) (separability implicit)	separable system ⇒ 1 action variable/cycle.

Examples:

→ H.O.: 1D
3D
→ general 1D
→ free particle in box

1) 1D H.O.

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)$$

$$\left(\frac{\partial S}{\partial q}\right)^2 + \omega^2 q^2 = E \quad \text{is H-J.}$$

$$I = \frac{1}{2\pi}\oint \left(E - \frac{\omega^2 q^2}{2}\right)^{1/2} dq$$

$$\oint = 2\int_{q_-}^{q_+} \qquad E = \omega^2 q^2 \rightarrow \text{turning pts.}$$

$$q_\pm = \pm\sqrt{E}/\omega$$

∴

$$I = \frac{2}{2\pi}\int_{q_-}^{q_+}\left[\left(E - \frac{\omega^2 q^2}{2}\right)\right]^{1/2} dq$$

$$q = \left(\sqrt{2E}/\omega\right)\sin\theta \quad , \quad dq = \frac{\sqrt{2E}}{\omega}\cos\theta$$

so, $I = E/\omega$

$P = I \equiv$ "new" momentum

$H = E = I\omega$ so, $\dot{\theta} = \dfrac{\partial H}{\partial I} = \omega$

$\theta = \omega t + \theta_0$

$$S = S(q, I) = \int_{q_0}^{q} dq \left(I\omega - \frac{\omega^2 q^2}{2}\right)^{1/2}$$

2) For 2D,

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{\omega_1^2 q_1^2}{2} + \frac{\omega_2^2 q_2^2}{2}$$

$H = f(1) + f(2) = E$ separable!

$f(1) = \dfrac{p_1^2}{2} + \dfrac{\omega^2 q^2}{2} = E_1 \rightarrow$ const.

$f(2) = \dfrac{p_2^2}{2} + \dfrac{\omega^2 q^2}{2} = E_2 \rightarrow$ const.

so, for action variables I_1, I_2:

$$I_1 = \frac{1}{2\pi}\oint p_1\, dq = \frac{1}{2\pi}\oint p_1(\xi_1)\, d\xi_1 = \frac{E_1}{\omega_1}$$

$$I_2 = E_2/\omega_2$$

$$H(I_1, I_2) = E = E_1 + E_2$$
$$= I_1\omega_1 + I_2\omega_2$$

→ separable, so
→ additive form of H in A-A variables

3) Free Particle in 2D $\begin{cases} 0 < x < a \\ 0 < y < b \end{cases}$
(hard wall)

$$H = \frac{1}{2m}(p_x^2 + p_y^2)$$

⇒ 2 Degs freedom ⇒ 2 I's, 2 θ's

$\therefore$ $$I_1 = \frac{1}{2\pi}\oint p_x\, dx$$

$$I_2 = \frac{1}{2\pi}\oint p_y\, dy$$

$$\oint p_x \, dx = \int_0^a p_{x_+} \, dx + \int_a^0 p_{x_-} \, dx$$

$p_{x_+} = - p_{x_-}$ (reverse when bounce off wall)

$$\oint p_x \, dx = 2a \, |p_x|$$

$$\therefore \quad I_1 = \frac{a}{\pi} |p_x|$$

$$I_2 = \frac{b}{\pi} |p_y|$$

So $$H = E = \frac{p_x^2 + p_y^2}{2m}$$

$$= \frac{\pi^2}{2m} \left(\frac{I_1^2}{a^2} + \frac{I_2^2}{b^2} \right)$$

$$\omega(I_{1,2}) = \frac{\partial E(I_1, I_2)}{\partial I_{1,2}} = \frac{\pi^2}{m} \frac{I_1}{a^2}, \; \frac{\pi^2}{m} \frac{I_2}{b^2}$$

2 points:

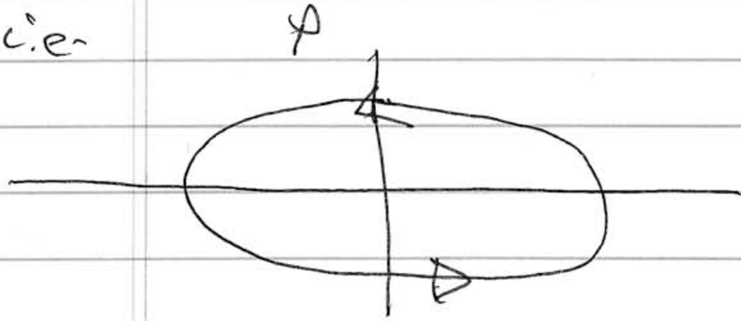
(a) contrast:

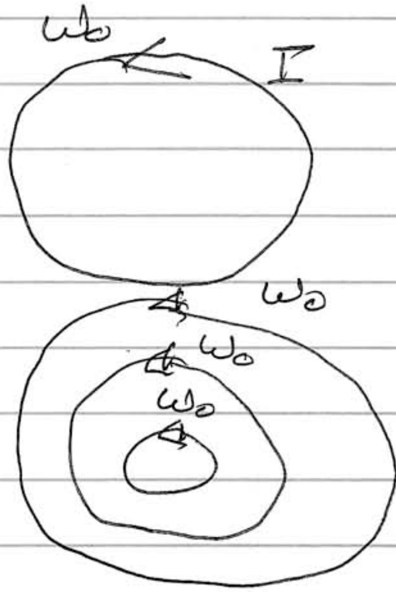
$\rightsquigarrow$ H.O. $\quad \omega(I) = \omega_0 = \text{const}$

$$\partial \omega / \partial I = 0$$

$I\omega_0 = E$ $\rightarrow$ constant frequency
$\rightarrow$ no shear in winding rate

i.e.





and all I contours have same rotation frequency ω_0

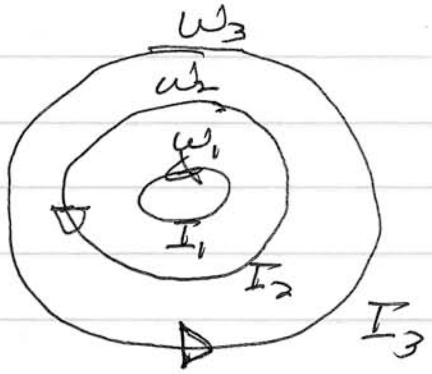
→ Box $\omega(I) = \frac{\pi^2}{ma^2} I$

$\partial\omega(I)/\partial I \neq$ $\omega \sim |p|$

→ winding rate varies with I

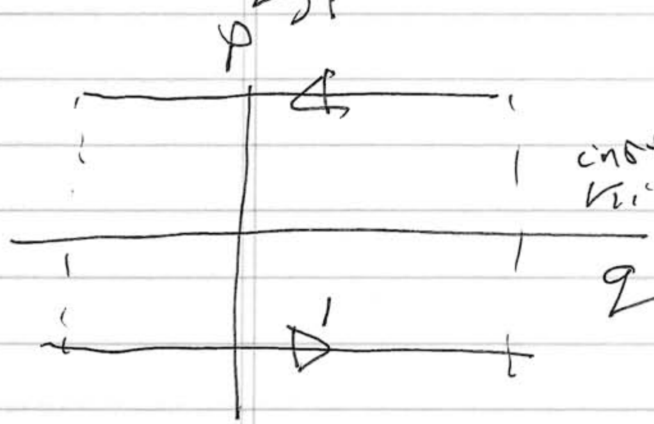
→ "shear"

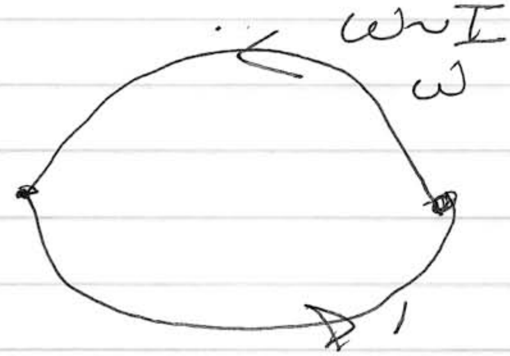
i.e



Winding rate increases with I (i.e. $\omega \sim I$) ⇒ differential rotation

q, p → I, ω





i.e. top box — top circle, etc.

? H.O. is linear problem, with $\partial\omega/\partial I = 0$

Box has $\partial\omega/\partial I \neq 0$, yet is linear too ?!

Why ?

n.b. Consider general 1D potential:

$$H = p^2 + V(q)$$

$$I = \oint \frac{p\,dq}{2\pi} = \frac{1}{2\pi}\oint \left[E - V(q)\right]^{1/2} dq$$

$$= \underline{I(E)}$$

$$\omega = \partial E(I)/\partial I$$

now, for $V(q) \sim \beta q^4$

$$I \sim c' E^{3/4}$$

$$\Rightarrow \quad E \sim c\, I^{4/3} \quad \text{so} \quad \omega(I) \sim c'' I^{1/3}$$

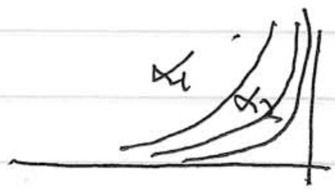
shear!

$\Rightarrow$ Nonlinearity develops from $V \sim \xi^{\alpha}$ potential for $\alpha > 2$.

$\therefore$ View hard wall as a limiting case

i.e.

$$V = V_0 \left(x/a \right)^{\alpha}$$



$\alpha \gg 1$

so hard wall boundary condition appears as nonlinearity due high high powers implicit in piecewise continuous potential.

② Reln. QM

classically: $H = E = \frac{\pi^2}{2m} \left(\frac{I_1^2}{a^2} + \frac{I_2^2}{b^2} \right)$

if $\left.\begin{array}{l} I_1 \to n\hbar \\ I_2 \to m\hbar \end{array}\right\}$ quantize action variables

$$E = \frac{\pi^2 \hbar^2}{2m} \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right) \rightarrow \text{eigenstates of free QM particle in box}$$

aside: Can observe correspondance

Classical	Quantum
$I = E/\omega$	$E = (N + 1/2)\hbar\omega$
$H = I\omega$	

N: # quanta (quantum #) $\leftrightarrow$ occupation

$\therefore$ suggests view I as classical # of excitations/waves $\leftrightarrow$ exciton density

straightfoward to generalize: (linear wave)

$$I = E/\omega \quad \longleftrightarrow \quad N(\underline{k}, \underline{x}) = E(\underline{k}, \underline{x})/\omega_k$$

$I = E/\omega$: linear H.O.

$E(\underline{k}, \underline{x})$: wave energy density

$N(\underline{k}, \underline{x})$: Action Density, as Wave Density, # waves

ω_k: wave frequency

→ General Properties of Motion in s dimensions.

Now, consider: system

- s degrees of freedom (arbitrary)
- separable H.-J. equation

$$S = \sum_{i=1}^{s} S_i(q_i)$$ (i.e. integrable)

∴ can define $\underline{s}$ action variables I_i

$$I_i = \oint \frac{p_i\,dq_i}{(2\pi)}$$ i.e. s - IOMs.

and $\Theta_i = \partial S_0/\partial I_i$ angle variables

so

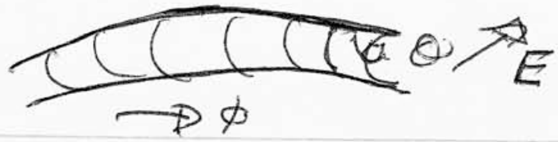
$$\dot{I}_i = 0$$

$$\Theta_i = \omega_i(E)\,t + t_0$$
$$\omega_i(E) = \partial E/\partial I_i$$

i.e. for $s = 2$

$$\dot{I}_1 = \dot{I}_2 = 0$$ $$\omega_{1,2} = \partial E/\partial I_{1,2}$$

$$\Theta_{1,2} = \omega_{1,2}(E)\,t + t_0$$



{ change I
→ change surf
→ nested surfaces.

∴ phase space is 2 torus. Fixed $E \Rightarrow$ motion on toroidal surface.
[In general, phase space is S-torus.]

$$\theta = \omega_1(E)\,t \qquad \theta = \frac{\omega_1(E)}{\omega_2(E)}\,\phi$$
$$\phi = \omega_2(E)\,t$$

→ now, for any $F(\underline{q}, \underline{p})$, can write:

fourier series

$$F = \sum_{l_1} \sum_{l_2} \cdots \sum_{l_s} A_{l_1, l_2 \ldots s} \exp\left[i(l_1\theta_1 + l_2\theta_2 + \cdots l_s\theta_s)\right]$$

$l_1, l_2, \ldots l_s$ integers. ⇒ define vector $\underline{l}$

equivalently:

$$\omega_i \;\S$$

$$F = \sum_{l_1} \sum_{l_2} \cdots \sum_{l_s} A_{l_1, l_2 \ldots l_s} \exp\left[it\left(\underline{l} \cdot \frac{\partial E}{\partial \underline{I}}\right)\right]$$

$$\underline{l} \cdot \frac{\partial E}{\partial I} = l_1 \frac{\partial E}{\partial I_1} + l_2 \frac{\partial E}{\partial I_2} + \cdots l_s \frac{\partial E}{\partial I_s}.$$

Now, in general:

→ frequencies not commensurate, so F not periodic i.e. $\underline{l} \cdot \partial E / \partial \underline{I}$ irrational

→ indeed, system generally not periodic in any coordinate (except for special E).

but, for sufficient time, system will come arbitrarily close to starting point. ⇒ [Poincare Recurrence Thm.!]

i.e. {trajectory ergodically covers surface of torus}

∴ motion is "conditionally" periodic.

But; degeneracy happens!

- degeneracy: $n\omega_i = m\omega_j$
- all s commensurate ⇒ complete degeneracy.

So, as in Kepler problem, ⇒ degeneracy implies reduction in number of independent I_i. Why?

Commensurate frequencies $\Rightarrow$

$$n_1 \omega_1 = n_2 \omega_2$$

$$n_1 \frac{\partial E}{\partial I_1} = n_2 \frac{\partial E}{\partial I_2}$$

so $E = E(n_2 I_1 + n_1 I_2)$

i.e. – energy depends on sum of action variables
linear superposition

$\Rightarrow$

– degeneracy

$\Rightarrow$

– can make canonical transformation
so $E = E(I')$, only.

$\Rightarrow$
$\therefore$ in degenerate motion, there is an increase in the number of one-valued integrals of the motion, relative to non-degenerate case.

i.e. non-degenerate motion – s degs freedom

$2s-1 \rightarrow$ IOM'S

$\left\{ \begin{array}{l} S \text{ values } I_i \Rightarrow \text{ single valued } I_i \\ s-1 \text{ values also } \Theta_i \, \partial E/\partial I_k - \Theta_k \, \partial E/\partial I_i \end{array} \right.$

note: $S-1$ values $\rightarrow$ phases (i.c.'s) of angle variables.

$\rightarrow$ not single valued.

but if degeneracy, note though:

$\rightarrow$ $n_1\theta_1 - n_2\theta_2$ not single valued (it is, to addition of 2π)

so

$\rightarrow$ $\sin(n_1\theta_1 - n_2\theta_2)$ is single valued.
(etc.)